

# SHAPE CONVERGENCE FOR AGGREGATE TILES IN CONFORMAL TILINGS

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ABSTRACT. Given a substitution tiling  $T$  of the plane with subdivision operator  $\tau$ , we study the conformal tilings  $\mathcal{T}_n$  associated with  $\tau^n T$ . We prove that aggregate tiles within  $\mathcal{T}_n$  converge in shape as  $n \rightarrow \infty$  to their associated Euclidean tiles in  $T$ .

## 1. INTRODUCTION

The term *tiling* refers to a locally finite decomposition of a topological plane into a pattern of compact regions known as its *tiles*. This paper involves tilings of four successive types: Starting from a *substitution* tiling, one can construct a *combinatorial* tiling, then an *affine* tiling, and finally a *conformal* tiling. Connections between the first and the last are the subject of this paper, with the middle two as necessary bridges. Here are the rough definitions of these objects.

- *Substitution tilings*  $T$ : This is a well-studied class of Euclidean tilings of the complex plane. Each tile in a substitution tiling  $T$  is similar to one of a finite set of polygonal prototiles. Moreover, there is an associated subdivision rule  $\tau$  specifying how each tile can be decomposed into a finite union of subtiles, each again similar to one of the prototiles. The tilings themselves are limits of successive subdivisions followed by rescaling.
- *Combinatorial tilings*  $K$ : These are abstract cell decompositions of a topological plane obtained by removing the metric properties from substitution tilings  $T$ . The geometric subdivision rule  $\tau$  for  $T$  becomes a combinatorial subdivision rule which can be applied to  $K$ .
- *Affine tilings*  $\mathcal{A}$ : These are obtained from combinatorial tilings  $K$  by identifying each  $n$ -sided cell of  $K$  with a unit-sided regular Euclidean  $n$ -gon. An affine tiling  $\mathcal{A}$  is not realized metrically in  $\mathbb{C}$ , but defines rather a plane with a piecewise Euclidean metric structure.
- *Conformal tilings*  $\mathcal{T}$ : These arise from affine tilings by imposing a canonical conformal structure in which each tile is conformally regular and enjoys a certain anticonformal *reflective property* across its edges. The resulting Riemann surface is conformally equivalent to  $\mathbb{C}$ , and its image under a conformal homeomorphism is what we refer to as a *conformal tiling*  $\mathcal{T}$ .

We use distinct symbols to distinguish these four categories and the symbol “ $\sim$ ” to denote corresponding objects. A substitution tiling  $T$  leads to a combinatorial tiling  $K$ , then to an affine tiling  $\mathcal{A}$ , and finally to a conformal tiling  $\mathcal{T}$ . Thus  $T \sim K \sim \mathcal{A} \sim \mathcal{T}$ . A tile  $t \in T$  is a Euclidean polygon, and corresponds to a tile  $k \in K$ , a combinatorial  $n$ -gon, which in turn corresponds to a tile  $a \in \mathcal{A}$ , a regular Euclidean  $n$ -gon, and this finally corresponds to a conformal tile  $t \in \mathcal{T}$ , which is a *conformal polygon*, that is, a topological polygon in the plane with analytic arcs as sides. Thus  $t \sim k \sim a \sim t$ . We will shortly review the definitions and properties of tilings, and in particular of conformal tilings as developed in [3].

A substitution tiling  $T$  comes with a subdivision operator  $\tau$ , and applying  $\tau$  leads to a new substitution tiling  $\tau T$  with the same set of prototiles. That subdivision operation is also inherited by the associated combinatorial tiling  $K \sim T$ , and we have  $\tau K \sim \tau T$ . In both these settings,  $\tau$  is considered an *in situ* operator, that is, it subdivides tiles in place within  $T$  or  $K$ . Figure 1 illustrates a fragment of the pinwheel tiling and its subdivision rule.

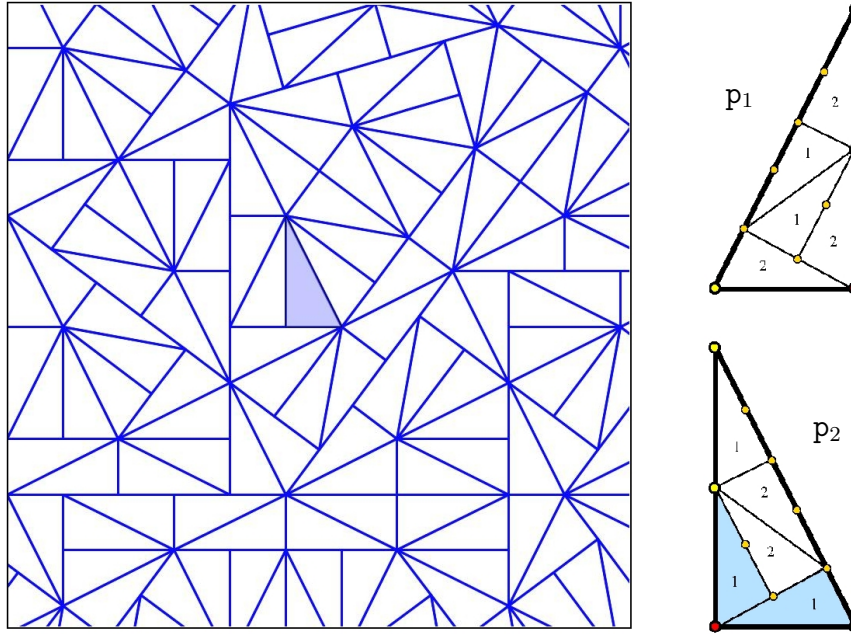


FIGURE 1. A fragment of a “pinwheel” substitution tiling of the plane, its two prototiles, and the subdivision rule  $\tau$ .

Things are different in the conformal setting. True, there is a conformal tiling  $\mathcal{T} \sim K$ , and a conformal tiling  $\mathcal{T}_1 \sim \tau K$ . It is not appropriate, however, to write  $\mathcal{T}_1 = \tau \mathcal{T}$ , since  $\tau$  does not generally act *in situ* in the conformal case. A conformal tile  $t$  is generally different in shape from its Euclidean counterpart  $t \sim t$ , and the unions of conformal tiles coming from successive subdivisions of  $t$  will have a succession of yet other shapes.

However, experiments in [3] (see §3.6) suggested that aggregate tiles — the union of tiles associated with successive subdivisions of a given tile — look increasingly like that tile’s Euclidean counterpart. In this paper we prove that this is indeed the case. Note in particular that the purely combinatorial tiling  $K \sim T$  and the combinatorial subdivision rule  $\tau$  somehow encode all the geometric information in  $T$  itself. This and other comments about our result will be discussed in Section 7.

## 2. EXAMPLE: THE PINWHEEL TILING

An early example may be helpful. The pinwheel tiling  $T$  was introduced by John H. Conway (see Charles Radin [10]). The tiles are all  $[1:2:\sqrt{5}]$  triangles. Due to orientation, there are two prototiles,  $\{p_1, p_2\}$ , pictured along with the associated subdivision rule  $\tau$  on the right in Figure 1. Note that  $\tau$  breaks each triangle into 5 similar triangles, their types indicated by “1” and “2”. The two shaded subtiles in  $p_2$  will be discussed shortly. Figure 2 focuses on root tile  $t$ , the shaded tile in Figure 1.

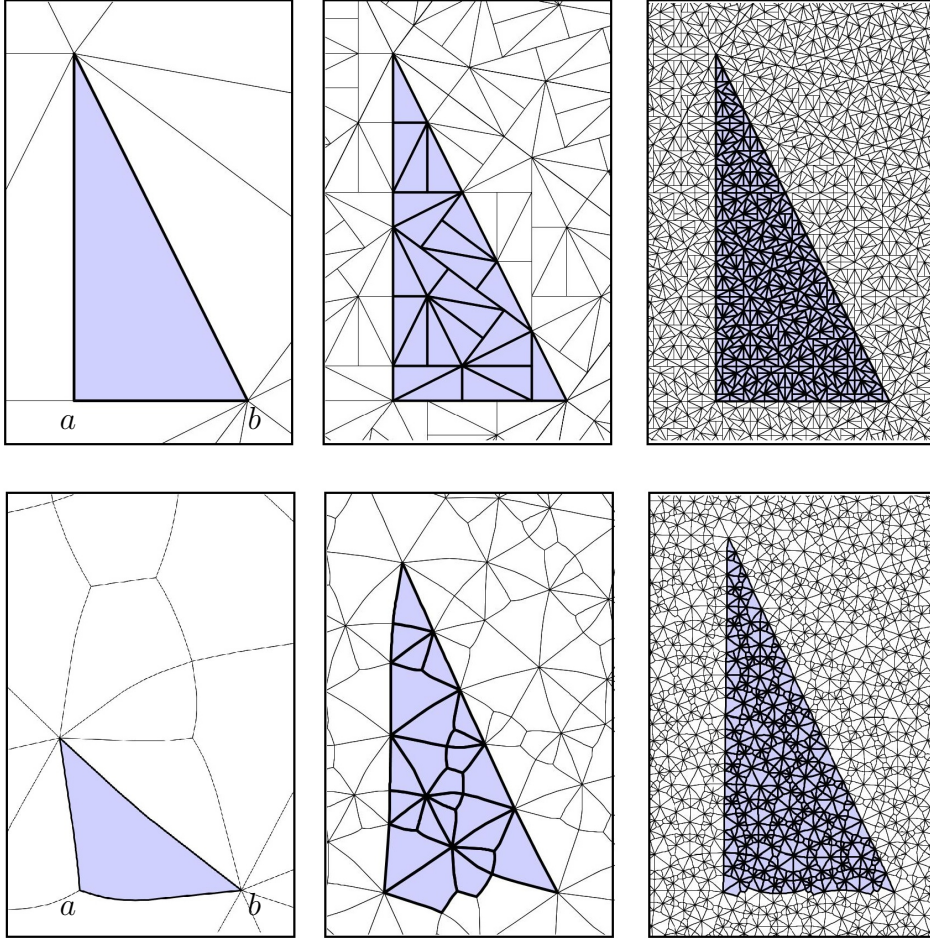


FIGURE 2. Across the top: Euclidean tile  $t$  and aggregates  $t^{(2)}$ ,  $t^{(4)}$ .  
Across the bottom: conformal versions  $\mathcal{t}$ ,  $\mathcal{t}^{(2)}$ ,  $\mathcal{t}^{(4)}$ .

The top row of Figure 2 shows  $T$ , the twice subdivided tiling  $\tau^2 T$ , and the four times subdivided tiling  $\tau^4 T$ . The subdivision operation occurs *in situ* in the Euclidean setting, so  $t$  is a union of its 25 subtiles in  $\tau^2 T$  and a union of its 625 subtiles in  $\tau^4 T$ .

The bottom row of Figure 2 shows the corresponding fragments of the associated conformal tilings, denoted  $\mathcal{T}$ ,  $\mathcal{T}_2$ , and  $\mathcal{T}_4$ , respectively. Each conformal version is normalized so the base corners are at 0 and 1, as in the top row. The tiles associated with  $t$  are again highlighted in blue. Note that subdivision in the conformal setting does not happen *in situ*: each subdivision of  $T$  engenders a new conformal structure. The 25 blue tiles in  $\mathcal{T}_2$  form what we call a 2-aggregate tile  $\mathcal{t}^{(2)}$ , while the 625 blue tiles from  $\mathcal{T}_4$  form a 4-aggregate tile  $\mathcal{t}^{(4)}$ .

The point of our paper is hinted at in the two tilings in the right column of Figure 2: although the subdivision action is not *in situ* in the bottom, the conformal  $n$ -aggregate tiles — aggregated after  $n$  stages of subdivision — seem to be converging in shape to the original Euclidean tile  $t$ . After some notation and definitions, we state a theorem which confirms this convergence.

### 3. TILING DETAILS

The substitution tilings  $T$  of interest here are *aperiodic*, *hierarchical* tilings of  $\mathbb{C}$  displaying *finite local complexity*. The most well known examples are the Penrose tilings, which will appear along with “chair”, “domino”, and “sphinx” examples in §7. The reader may refer to [3] for further background. Briefly, each substitution tiling  $T$  has an associated finite set  $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_q\}$  of Euclidean polygonal *prototiles*, with every tile  $t \in T$  being similar to one of the prototiles. We note that prototiles are distinguished by shape, orientation, a designated base edge  $\langle a, b \rangle$ , and possibly by some abstract “label”. By a “similarity” of tiles, we mean a geometric similarity as polygons that also respects these features. Every prototile has an associated decomposition into subtiles, each subtile again being similar to one of the prototiles. These decompositions define a *subdivision* operator  $\tau$  which, when applied to  $T$  yields a new substitution tiling  $\tau T$  with the same prototiles. In the traditional theory, subdivision may be accompanied by an associated renormalization that is, (rescaling); that will not be the case here, so  $\tau$  operates as an *in situ* subdivision operator.

As noted earlier, if we ignore the metric properties of  $T$ , we are left with a cell decomposition of a topological plane, which we treat as a combinatorial tiling and denote by  $K$ . To avoid ambiguity, we require a condition of  $K$ : any two of its tiles are either disjoint or their intersection is a union of vertices and/or full edges. Note in the pinwheel subdivision rule of Figure 1, for example, that although the tiles are Euclidean triangles, the pattern of tile intersections requires four designated corners (marked by dots), and pinwheel tiles are thus considered 4-sided. The faces of  $K$ , which are combinatorial polygons, retain their associations with prototiles so that  $\tau$

may, abusing notation, be treated as a combinatorial subdivision operator: thus  $K \sim T$  implies  $\tau K \sim \tau T$ .

With  $K$  in hand, we consider the associated affine tiling  $\mathcal{A}$ . Every tile  $k \in K$  may be identified with a unit sided regular equilateral  $n$ -gon  $p$ , where  $n$  is the number of sides of  $k$ . If tiles  $k_1, k_2$  share an edge, then their polygons  $p_1, p_2$  are identified isometrically along the corresponding edge. This allows us to define a piecewise Euclidean metric on  $K$  with conical singularities. Thus we arrive at our affine tiling  $\mathcal{A}$ , writing  $\mathcal{A} \sim K$ . Note that the corners of tiles of  $\mathcal{A}$  are generically non-flat cone points, so we do not attempt to realize  $\mathcal{A}$  in any concrete setting.

This brings us to the tiling of direct interest, the conformal tiling  $\mathcal{T}$ . There is a canonical conformal atlas on  $\mathcal{A}$ : interiors of the regular polygons provide charts for their points, interiors of unions of two adjacent regular polygons provide charts for interior points of tile edges, and local power maps provide charts for tile vertices. With this atlas, the tiling  $\mathcal{A}$  becomes a Riemann surface. Since  $\mathcal{A}$  is simply connected and not compact, the classical Uniformization Theorem in complex analysis implies existence of a conformal homeomorphism  $\phi : \mathcal{A} \rightarrow \mathbb{G}$  where  $\mathbb{G}$  is either the unit disc  $\mathbb{D}$  or the complex plane  $\mathbb{C}$ .

**Definition.** *Given an affine tiling  $\mathcal{A}$ , the images  $\{\phi(a) : a \in \mathcal{A}\}$  under a conformal homeomorphism  $\phi : \mathcal{A} \rightarrow \mathbb{G}$  form a **conformal tiling** in  $\mathbb{G}$ . We write  $\mathcal{T}$  for this tiling and note that it is uniquely determined up to Möbius transformations of  $\mathbb{G}$ .*

In a sense,  $\mathcal{A}$  with its conformal structure is already a conformal tiling. However, under the map  $\phi$ , the individual tiles become concrete shapes in  $\mathbb{G}$ , and it is these shapes which are of interest in our work. The properties of  $\mathcal{T}$  and its tiles are developed fully in [3]. We need not be concerned with details, but some features are noteworthy: Each conformal tile  $t \in \mathcal{T}$  is a *conformal polygon*, a curvilinear polygon with sides which are analytic arcs. Indeed, each is a *conformally regular polygon*, meaning that there is a conformal self-map  $f : t \rightarrow t$  which maps each corner to the next;  $f$  has a single fixed point, the *conformal center* of  $t$ . Conformal tiles  $t, t'$  sharing an edge have a anti-conformal reflective relationship across that edge, which leads to an important rigidity phenomenon in conformal tilings: the shape of any single tile of  $\mathcal{T}$  determines uniquely the shapes and locations of every other tile of  $\mathcal{T}$ .

There remains the issue of whether  $\mathcal{T}$  lies in  $\mathbb{D}$  or  $\mathbb{C}$ . This is known as the “type” problem and might normally require some work to resolve. For substitution tilings of the plane with finite local complexity, however, the associated conformal tilings are always *parabolic*, that is,  $\mathbb{G} = \mathbb{C}$ . This will follow from the quasiconformal arguments in Lemma 6.1 below.

#### 4. SUBDIVISION DETAILS

Our substitution tilings  $T$  are assumed to display *finite local complexity*. This means simply that any two tiles can be juxtaposed in at most finitely many ways, up to similarity. The same then holds for any subdivision  $\tau^n T$ . We must place a

side condition on  $T$ , one that is nearly universal, failing in only the most trivial of substitution tilings. Nevertheless, we make it explicit for later use.

**Standing Assumption on  $T$ :** *There must exist a configuration  $\mathcal{C} = p \cup q$ , a union of two Euclidean tiles, so that the following holds:*

- (1) *both  $p$  and  $q$  are congruent to the same prototile  $\mathbf{p}$ ;*
- (2) *if  $S : p \rightarrow q$  is a congruence, then the linear part of  $S$  is not  $\pm I$ , plus or minus the identity.*
- (3) *for in every open disc  $D = D(r, z) \subset \mathbb{C}$  there exists an integer  $n$  so that  $\tau^n T$  contains a pair of tiles,  $t_1, t_2$  having the type of  $\mathbf{p}$  so that their union  $t_1 \cup t_2$  is similar to  $\mathcal{C}$ .*

In the tile schematics of Figure 1, the union of the two shaded tiles is an example of such a configuration  $\mathcal{C}$ : both tiles are congruent to  $\mathbf{p}_1$  and are not translations of one another. A similar configuration will occur in the subdivision of every tile similar to  $\mathbf{p}_2$  and thus will occur densely throughout the plane as  $T$  undergoes subdivision. In general, of course, the tiles forming  $\mathcal{C}$  are not necessarily contiguous or subtiles of the same parent.

## 5. STATEMENT OF THE THEOREM

A substitution tiling  $T$  is completely compatible with its subdivision operator  $\tau$  in that  $\tau T$  is an *in situ* decomposition of  $T$  into subtiles. More generally, for every positive integer  $n$ ,  $\tau^n T$  is an *in situ* decomposition of  $\tau^{n-1} T$ , and hence by induction, of  $T$  itself. For convenience we write  $T_n$  for  $\tau^n T$ . And if  $K \sim T$ , then we write  $K_n = \tau^n K$ , noting that  $K_n \sim T_n$ .

The reverse of subdivision is *aggregation*. Fix  $n \geq 0$  and focus on a combinatorial tile  $\mathbf{k} \in K_n$ . Now perform an additional  $m$  subdivisions of  $K_n$  to get  $K_{n+m}$ . Write  $\mathbf{k}^{(m)}$  for the union of tiles in  $K_{n+m}$  which were generated during the  $m$  subdivisions of  $\mathbf{k}$ : this union  $\mathbf{k}^{(m)}$  will be called an *m-aggregate tile* in  $K_{n+m}$  and the combinatorial connection to  $\mathbf{k}$  is indicated by writing  $\mathbf{k}^{(m)} \sim \mathbf{k}$ . In other words, an *m-aggregate tile* is a union of tiles at the  $(n + m)$ th subdivision stage which form a single tile  $\mathbf{k}$  from the  $n$ th subdivision stage.

This notion of aggregation (and the associated notations) apply equally to substitution, combinatorial, affine, and conformal tilings. However, the geometric differences are the subject of this paper. Consider a substitution tiling  $T$ , a tile  $t \in T_n$ , the combinatorial tile  $\mathbf{k} \sim t$  in  $K_n$ , and its *m-aggregate*  $\mathbf{k}^{(m)}$ . By definition,  $\mathbf{k}^{(m)} = \mathbf{k}$ . Likewise, because subdivision occurs *in situ* for substitution tilings, the *m-aggregate tile*  $t^{(m)} \sim \mathbf{k}^{(m)}$  is equal as a point set to the original tile  $t \in T_n$ .

This is not the case for the corresponding conformal tiles. Let  $\mathcal{T}$ ,  $\mathcal{T}_n$ , and  $\mathcal{T}_{n+m}$  be conformal tilings, where  $\mathcal{T} \sim K$ ,  $\mathcal{T}_n \sim K_n$ , and  $\mathcal{T}_{n+m} \sim K_{n+m}$ ,  $n, m \geq 0$ . Consider  $t \in \mathcal{T}_n$ , with  $t \sim \mathbf{k}$ . In general, the union of conformal tiles of  $\mathcal{T}_{n+m}$  corresponding to the *m-aggregate tile*  $\mathbf{k}^{(m)}$  is *not equal* as a point set to the conformal tile  $t \in \mathcal{T}_n$ . This

can be seen in the conformal images of the bottom row in Figure 2: the 2-aggregate  $t^{(2)}$  shown in the middle image, a union of 25 blue conformal tiles, has a shape unequal to its parent conformal tile, shown in the image to its left. Likewise, the 4-aggregate  $t^{(4)}$  shown on the right has yet another shape. Comparing the right side images from the top and bottom rows, however, the shapes appear to be getting close. Before our statement, we need to formalize a notion of shape.

**Definition.** *Jordan domains  $\Omega_j$  in the plane are said to **converge in shape** to a Jordan domain  $\Omega$  if there exist Euclidean similarities  $\Lambda_j$  of the plane so that  $\Lambda_j(\Omega_j)$  converges to  $\Omega$  in the Hausdorff metric as  $j \rightarrow \infty$ . Write  $\Omega_j \xrightarrow{\text{shape}} \Omega$ .*

Conformal tilings can be approximated in practice using methods of circle packing. Experiments carried out by the third author and Phil Bowers in [3] suggested that the phenomenon of shape evolution seen with the pinwheel tiling is not unique. A peek ahead to Figure 3 reveals shape comparisons for some other well known substitution tilings. The experiments that gave us these images led to the shape question ([3, page 38]) which is answered affirmatively here.

**Theorem 5.1.** *Let  $T$  be a substitution tiling of the plane with subdivision rule  $\tau$  and let  $\mathcal{T}_n$  denote the conformal tilings associated with  $\tau^n T$ . Fix a tile  $t \in T$ . For each  $n \geq 0$  let  $t^{(n)}$  be the associated  $n$ -aggregate tile in  $\mathcal{T}_n$ . Then  $t^{(n)} \xrightarrow{\text{shape}} t$  as  $n \rightarrow \infty$ .*

## 6. PROOF

We will work with various homeomorphisms between tilings. These will be termed *tiling maps*, in that they carry each tile of the domain tiling bijectively to the associated tile of the range tiling. Each tile is identified with some prototile, and tiling maps are always assumed to respect tile types and designated base edges  $\langle a, b \rangle$ .

We are free to normalize our tilings with similarities, so assume that  $T$  is positioned so that the tile  $t$  of interest has two of its corners  $a, b$  at 0, 1, respectively. Subdivision of  $T$  occurs in place, so these corners of the aggregate tiles  $t^{(n)} \subset T_n$  remain at 0 and 1. On the conformal side, for each  $n$  we apply a similarity to put the corresponding corners of the aggregate conformal tile  $t^{(n)} \sim t^{(n)}$  at 0 and 1 as well. Quasiconformal maps  $F_n : T_n \rightarrow \mathcal{T}_n$  are key to our proof.

**Lemma 6.1.** *Let  $T$  be a substitution tiling and  $\mathcal{T}$  the associated conformal tiling. Then there exists a  $\kappa$ -quasiconformal tiling map  $f : T \rightarrow \mathcal{T}$ , where  $\kappa = \kappa(T)$  depends only on the finite set of prototiles for  $T$ . In particular,  $\mathcal{T}$  is parabolic.*

*Proof.* We define  $f$  via an intermediate tiling map  $g : T \rightarrow \mathcal{A}$ , which is constructed tile-by-tile based on prototile type.

Consider a prototile  $p$  having  $m$  vertices. Define in *ad hoc* fashion a straight-edge triangulation of  $p$  by adding a finite number of vertices and Euclidean line segments as necessary in the interior of  $p$ . Any tile  $a \in \mathcal{A}$  having the type of  $p$  is a regular Euclidean  $m$ -gon. Define a straight-edge triangulation of  $a$  in the pattern of the triangulation of

$\mathbf{p}$ , with corners going to corresponding corners. This can be easily done, for example, by a Tutte embedding [11].

The upshot is that we have decomposed  $\mathbf{p}$  and  $\mathbf{a}$  into combinatorially equivalent patterns of Euclidean triangles. If  $t \in T$  has the type of  $\mathbf{p}$ , we may transfer the triangulation of  $\mathbf{p}$  by similarity to  $t$ , respecting corner designations. Now, define a continuous map  $g_t : t \rightarrow \mathbf{a}$  by mapping each of these triangles of  $t$  affinely onto the corresponding triangle of  $\mathbf{a}$ . Affine maps are quasiconformal and we may take  $\kappa$  to be the maximum of dilatations not only for the finitely many triangles of  $\mathbf{p}$ , but for the finite number of triangles over all prototile types. Thus  $g_t$  is  $\kappa$ -quasiconformal on the interior of  $t$ .

Applying the construction to every tile of  $T$  and noting that when tiles  $t, t'$  share an edge  $e$ , the maps  $g_t, g_{t'}$ , being affine, agree on  $e$ , we obtain a continuous map  $g : T \rightarrow \mathcal{A}$ . Since  $g$  is  $\kappa$ -quasiconformal on each tile and the union of boundaries of the triangles has area zero,  $g$  is  $\kappa$ -quasiconformal on all of  $T$ .

Recall that we defined a conformal structure on  $\mathcal{A}$  which is compatible with its p.w. affine structure and a conformal map  $\phi : \mathcal{A} \rightarrow \mathcal{T}$ . The map  $f : T \rightarrow \mathcal{T}$  is defined by  $f = \phi \circ g$  and, because  $\phi$  is 1-quasiconformal,  $f$  is  $\kappa$ -quasiconformal.

Finally, as  $T$  fills  $\mathbb{C}$ , Liouville's theorem for quasiconformal mapping [9] implies that the image  $\mathcal{T}$  fills  $\mathbb{C}$  as well. So  $\mathcal{T}$  is parabolic.  $\square$

**6.1. Reduction.** Observe that for any  $n \geq 0$  we may define the  $\kappa$ -quasiconformal map  $F_n : T_n \rightarrow \mathcal{T}_n$  in the fashion of Lemma 6.1 and that we have a uniform  $\kappa$ , since  $\kappa(T) = \kappa(T_n)$  for all  $n \geq 0$ . Since each  $F_n$  fixes the points 0 and 1 by our earlier normalization, it is a normal family: there exists a subsequence  $\{F_{n_j}\}$  which converges to a  $\kappa$ -quasiconformal limit function  $F$  which also fixes 0, 1: thus  $F_{n_j} \rightarrow F$  uniformly on compacta in  $\mathbb{C}$  as  $n_j \rightarrow \infty$ . We will prove that  $F$  is the identity map. Then, since the aggregate conformal tiles  $t^{(n)} = F_n(t^{(n)})$  converge pointwise to  $F(t^{(n)})$  and since  $t^{(n)} = t$ , the fact that  $F(t) = t$  will imply the desired conclusion,  $t^{(n)} \xrightarrow{\text{shape}} t$ . In hindsight we may observe that in fact the full sequence  $\{F_n\}$  converges to the identity.

To prove that  $F$  is the identity, recall that as a  $\kappa$ -quasiconformal map it has a derivative  $dF(z)$  for almost every (with respect to Lebesgue measure)  $z \in \mathbb{C}$ . If  $dF(z)$  exists and is a similarity, then the dilatation of  $F$  at  $z$  must be 1. If the dilatation is 1 a.e., then  $F$  is 1-quasiconformal — that is,  $F$  is an entire function. Since  $F$  is a homeomorphism fixing 0 and 1, we can conclude that  $F$  is the identity.

To complete our proof, therefore, it is enough to show that the linear mapping  $L_z$  associated with  $dF(z)$  is a similarity for almost all  $z \in \mathbb{C}$ . We fix attention on a point  $z_0$  where  $dF(z_0)$  exists. Translating  $T$  by  $-z_0$  and the mappings  $F_n$  by  $-F_n(z_0)$ , we may assume without loss of generality that  $z_0 = 0$  and  $F_n(0) = F(0) = 0$ . The remainder of the proof then consists in proving this

**Claim 6.2.**  $L = L_0$  is a similarity.



**6.2. Proof of the Claim.** A bit of notation first. We currently have extracted the subsequence  $\{F_{n_j}\}$  with

$$(A) \quad F_{n_j} \rightarrow F \text{ uniformly on compacta as } n_j \rightarrow \infty.$$

A finite number of further extractions will be needed, so we abuse notation by referring in each instance to  $\{n_j\}$  as the latest subsequence. We will also frequently use index “ $n$ ” when referring to “ $n_j$ ”; context should make our intentions clear.

To zoom in at 0, we introduce *blowups*. Given  $\psi : \mathbb{C} \rightarrow \mathbb{C}$  with  $\psi(0) = 0$  and given  $\rho > 0$ , define  $\psi^\rho(z) = \psi(\rho z)/\rho$ ,  $z \in \mathbb{C}$ . Note that  $\psi^\rho(0) = 0$  and if the differential  $d\psi(0)$  exists, then  $d\psi(0) = d\psi^\rho(0)$ , and the limit as  $\rho \rightarrow 0$  is a linear map:

$$\lim_{\rho \rightarrow 0} \psi^\rho(z) = [d\psi(0)](z).$$

The blowups to consider are  $F_n^\rho$  and  $F^\rho$ . Observe that  $dF^\rho(0) = dF(0)$  for all  $\rho$ , implying that

$$(B) \quad F^\rho \longrightarrow L \text{ uniformly on compacta of } D \text{ as } \rho \downarrow 0.$$

Moreover, due to (A), for each fixed  $\rho > 0$  and  $n \geq 0$ ,

$$(C) \quad F_{n+m}^\rho \rightarrow F^\rho \text{ uniformly on compacta as } m \rightarrow \infty.$$

(Note an additional abuse of notation here: As  $n$  denotes  $n_j$ , so  $n + m$  here denotes  $n_{j+m}$  so that we may appeal to (A). We will assume this meaning without further comment when we apply (C).)

**6.2.1. In the Domain.** We may henceforth restrict attention to some disc  $D$  centered at 0 (e.g.,  $D = \mathbb{D}$ ) as the common domain. The special configuration  $\mathcal{C}$  defined in §4 is a union of two congruent copies of some prototile  $\mathbf{p}$ . A copy of  $\mathcal{C}$  among the tiles of  $T_n$  occurs in arbitrarily small neighborhoods of 0 for sufficiently large  $n$ . Therefore, we can extract a (further) subsequence  $\{F_{n_j}\}$  and a corresponding sequence  $\rho_{n_j}$  of blowup parameters,  $\rho_{n_j} \downarrow 0$ , so that the scaled tilings  $T_{n_j}/\rho_{n_j}$  contain configurations  $\mathcal{C}_{n_j} \subset D$  similar to  $\mathcal{C}$  and with diameters bounded above and below. By compactness in the Hausdorff metric we may extract a further subsequence  $\{n_j\}$  so that  $\mathcal{C}_{n_j} \rightarrow \tilde{\mathcal{C}}$  where the limit configuration  $\tilde{\mathcal{C}} \subset D$  is again similar to  $\mathcal{C}$ .

For each  $n = n_j$ , the configuration  $\mathcal{C}_n$  is a union  $p_n \cup q_n$  of scaled tiles from  $T_n/\rho_n$ . Here  $p_n$  and  $q_n$  are congruent *via* a similarity  $S_n : p_n \rightarrow q_n$  that is not a translation. Likewise, in the limit  $\tilde{\mathcal{C}}$  is a union of polygons  $p$  and  $q$  that are congruent to one another *via* a similarity  $S : p \rightarrow q$  that is not a translation. We note the following convergence properties for later use:

$$(D) \quad p_n \rightarrow p, \quad q_n \rightarrow q, \quad \text{and } S_n \rightarrow S, \quad \text{as } n = n_j \rightarrow \infty.$$

6.2.2. *In the Range.* In the range we have several additional sets to consider. Let  $\tilde{p}, \tilde{q}$  be the Euclidean polygons  $\tilde{p} = L(p)$  and  $\tilde{q} = L(q)$  and define the affine transformation  $\tilde{S} : \tilde{p} \rightarrow \tilde{q}$  by  $\tilde{S} = L \circ S \circ L^{-1}$ . Note that, as usual,  $\tilde{S} : \tilde{p} \rightarrow \tilde{q}$  identifies the distinguished corners of  $\tilde{p}$  and  $\tilde{q}$ . The conclusion will follow once we show that  $\tilde{S}$  is conformal.

Consider the images of aggregates associated with the tiles  $p_n, q_n$ . For each  $m \geq 0$ , the tiles  $p_n$  and  $q_n$  in  $D$  may be subdivided  $m$  times by  $\tau$  to give the  $m$ -aggregate tiles  $p_n^{(m)} = \tau^m p_n$  and  $q_n^{(m)} = \tau^m q_n$  within  $T_{n+m}/\rho_n$ . Of course, since this subdivision takes place *in situ* in the domain,  $p_n^{(m)} = p_n$  and  $q_n^{(m)} = q_n$  as point sets in  $D$ . However, the additional  $m$  subdivisions have implications for the conformal images in the range. We denote the aggregate images by

$$\begin{aligned}\tilde{p}_{n,m} &= F_{n+m}^{\rho_n}(p_n^{(m)}) = F_{n+m}^{\rho_n}(p_n), \\ \tilde{q}_{n,m} &= F_{n+m}^{\rho_n}(q_n^{(m)}) = F_{n+m}^{\rho_n}(q_n).\end{aligned}$$

The last bit of tile notation is this:

$$\hat{p}_n = F^{\rho_n}(p_n) \quad \text{and} \quad \hat{q}_n = F^{\rho_n}(q_n).$$

Using (C) we have:

$$(E) \quad \tilde{p}_{n,m} \rightarrow \hat{p}_n, \quad \tilde{q}_{n,m} \rightarrow \hat{q}_n, \quad \text{as } m \rightarrow \infty.$$

In the range we define the homeomorphisms  $\Phi_{n,m} : \tilde{p}_{n,m} \rightarrow \tilde{q}_{n,m}$  by

$$\Phi_{n,m} = F_{n+m}^{\rho_n} \circ S_n \circ (F_{n+m}^{\rho_n})^{-1}.$$

We claim that  $\Phi_{n,m}$  is in fact conformal. Recall that the quasiconformal maps  $F_{n+m}$  were defined in a tile-by-tile fashion. The action on each tile  $t$  having the type of, say,  $p_j$ , was modeled on a composition  $f_j \circ g_j$ , where  $g_j$  is a piecewise affine map carrying some fixed triangulation of  $p_j$  to an equivalent triangulation of a regular Euclidean  $n$ -gon and  $f_j$  is a conformal map of that  $n$ -gon to the conformal image tile. But each subtile of  $\tilde{p}_{n,m}$  is mapped by  $\Phi_{n,m}$  to a subtile of  $\tilde{q}_{n,m}$  sharing the same tile type, so the composition defining  $\Phi_{n,m}$  is modeled on compositions of the form

$$f_j \circ g_j \circ S \circ g_j^{-1} \circ f_j^{-1}.$$

Since  $S$  is a similarity and the  $g_j$  is canonical for the given tile type, the center three factors give a similarity. Since  $f_j$  is conformal, this means that the restriction of  $\Phi_{n,m}$  to each subtile of  $\tilde{p}_{n,m}$  is conformal. The edges between subtiles form a set of area zero, so  $\Phi_{n,m} : \tilde{p}_{n,m} \rightarrow \tilde{q}_{n,m}$  is a conformal mapping.

Noting the convergence in (E) and applying normal families and the Caratheodory Kernel Theorem from conformal function theory, we obtain a limit conformal mapping  $\Phi_n : \hat{p}_n \rightarrow \hat{q}_n$ . More explicitly, recalling the definition of  $\Phi_{n,m}$  and the fact that  $p_{n,m} = p_n$  and  $q_{n,m} = q_n$ , we have

$$\Phi_n = F^{\rho_n} \circ S_n \circ (F^{\rho_n})^{-1}, \quad \Phi_n : \hat{p}_n \xrightarrow{(F^{\rho_n})^{-1}} p_n \xrightarrow{S_n} q_n \xrightarrow{F^{\rho_n}} \hat{q}_n.$$

6.2.3. *Conclusion.* To conclude the proof of the claim, note that as we let  $n = n_j$  go to infinity, we have  $\rho_n$  going to zero as well. Using (B) and (D) we see that  $\hat{p}_n \rightarrow \tilde{p}$  and  $\hat{p}_n \rightarrow \tilde{p}$ . The sequence  $\{\Phi_n\}$  is a normal family, and so up to taking a subsequence the  $\Phi_n$  converge to a limit conformal mapping  $\Phi$ ,

$$\Phi_n \rightarrow \Phi : \tilde{p} \rightarrow \tilde{q}$$

Since by (B) the blowups  $F^{\rho_n}$  converge uniformly on compacta of  $D$  to the linear transformation  $L$ , we see that  $\Phi = L \circ S \circ L^{-1}$  on  $\tilde{p}$ . In particular,  $L \circ S \circ L^{-1}$  is a similarity. Since  $S$  is a similarity whose linear part is not  $\pm I$ , then the fact that  $L \circ S \circ L^{-1}$  is a similarity implies  $L$  must be a similarity. This completes the proof of the Claim and hence of the Theorem.

## 7. EXAMPLES AND QUESTIONS

Figure 3 illustrates four substitution tilings from the traditional tiling literature: the “chair”, “domino”, “sphinx”, and “Penrose”. The Euclidean shapes are shown with their subdivision schemes. Note that the Penrose tile here is a “dart” from the familiar “kite/dart” version of Penrose tilings; in actuality, there are four Robinson prototiles involved, and all four appear in the kite subdivision.

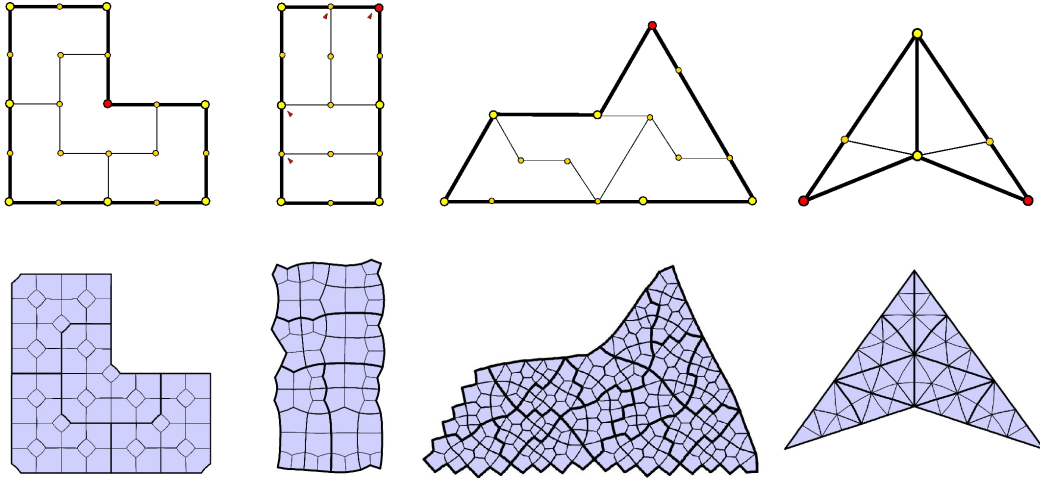


FIGURE 3. Along the top: chair, domino, sphinx, and Penrose Euclidean tiles and their subdivisions. Along the bottom, associated aggregate tiles isolated from conformal tilings. The two on the left are 3-aggregates, the remaining two, 4-aggregates.

An intriguing aspect of shape convergence is that the purely combinatorial data in  $K$  and  $\tau$  encode precise Euclidean shapes. Thus the combinatorics of the pinwheel knows about  $\sqrt{5}$ , those of the domino know the aspect ratio 1:2, the Penrose kite knows the golden ratio hidden throughout its tilings. This connection between combinatorics

and geometry has many precedents, of course. The most farreaching, perhaps, is in Gorthendieck’s *dessins d’Enfants*, wherein abstract finite graphs (“child drawings”) lead to algebraic number fields (see [8, 2]). Another deep connection is proposed in Cannon’s Conjecture, [4, 6, 7], concerned with the recognition of Klienian groups from the combinatorics of subdivision operators. In conformal tiling itself, the foundational example was the pentagonal tiling studied in [1]. Jim Cannon, Bill Floyd, and Walter Parry, along with the first author, Rick Kenyon, proved in [5] that it arises from iteration of the inverse of a rational function with integer coefficients, leading, for example, to the wonderful scaling factor of the tiling,  $\lambda = (324)^{-1/5}$ .

In this broader view, the combinatorics of substitution tilings are rather rarified — examples are difficult to come by and the handful available are prized. In contrast, the combinatorics of conformal tilings can be nearly arbitrary, even if one specifies finite numbers of tile types and finite local complexity. Several examples in [3] not associated with traditional substitution tilings illustrate this ubiquity. Typically the finite number of combinatorial tile types have, in their conformal tilings, infinitely many Euclidean shapes. Although general limiting behaviours of aggregate tiles have not yet been studied closely in the conformal setting, [3] does identify an important class of “conformal” subdivision rules  $\tau$ , under which tile shapes, though infinite in variety, still subdivide *in situ*.

This landscape of combinatorial/geometric interactions raises some natural questions which we pose to the interested reader.

**Question 1.** *Are there criteria to determine whether a given finite collection of combinatorial tile types with a combinatorial subdivision operator  $\tau$  is associated with a substitution tiling?*

**Question 2.** *Can one discover new substitution tilings in this way?*

**Question 3.** *What is the limiting fate of aggregate conformal tiles for general combinatorial subdivision tilings?*

A last comment is about the experiments behind this paper: Circle packing is as yet the only method for approximating conformal tilings in practice. The examples illustrated here were created in the software package **CirclePack**. This software is available on the third author’s web site. In addition, the **CirclePack** scripts for creating and manipulating the specific examples in the paper are available from the third author on request.

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